PDEs and their Equivalent Formulations

1. Elliptic Partial Differential Equations (diffusion-reaction)

- $$\begin{cases} -\nabla\cdot(A\nabla u) + a u = f & \text{in } \Omega \subset R^d, \\ u\big|_{\Gamma_D} = g_D \text{ and } \vec{n}\cdot(A\nabla u)\big|_{\Gamma_N} = g_N, \end{cases}$$

where (1) Ω is the domain with boundary

$$\partial\Omega = \Gamma_D \cup \Gamma_N \quad \text{s.t.} \quad \Gamma_D \cap \Gamma_N = \phi;$$

(2) coefficients: $A = (a_{ij})_{d\times d}$ is diffusion coefficient

a is reaction coefficient

(3) f, g_D, and g_N are given scalar-valued functions defined on the domain Ω, Dirichlet boundary Γ_D, and Neumann boundary, respectively,

(4) $\vec{n}$ is the unit outward vector normal to the boundary.

(5) ∇ is the gradient operator defined by

$$\nabla u = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_d} \end{pmatrix}$$

(6) $\nabla\cdot$ is the divergence operator defined by

$$\nabla\cdot\vec{v} = \begin{pmatrix}\frac{\partial}{\partial x_1}\\ \vdots \\ \frac{\partial}{\partial x_d}\end{pmatrix}\cdot\begin{pmatrix}v_1\\ \vdots \\ v_d\end{pmatrix} = \frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_d}{\partial x_d}.$$

- <u>Formula of Integration by Part</u>

$$\int_\Omega (\nabla\cdot\vec{u})\, v\, dx = -\int_\Omega \vec{u}\cdot\nabla v\, dx + \int_{\partial\Omega} (\vec{u}\cdot\vec{n})\, v\, ds$$

- <u>Variational (weak) Formulation</u>

$$\int_\Omega f v\, dx = \int_\Omega \left[-\nabla\cdot(A u) + a u\right] v\, dx$$

$$= \int_\Omega \left[A\nabla u\cdot\nabla v + a u v\right] dx - \int_{\Gamma_N} g_N v\, ds - \int_{\Gamma_D} [\vec{n}\cdot A\nabla u]\, v\, ds$$

if $v|_{\Gamma_D} = 0$, then

$$\boxed{\int_\Omega \left[A\nabla u\cdot\nabla v + a u v\right] dx = \int_\Omega f v\, dx + \int_{\Gamma_N} g_N v\, ds}$$

for any v

(1) Solution space

$$H^1(\Omega) = \left\{ v \in L^2(\Omega) \;\middle|\; \int_\Omega \left[v^2 + |\nabla v|^2 \right] dx < +\infty \right\}$$

$$u \in H^1_g(\Omega) = \left\{ v \in H^1(\Omega) \;\middle|\; v|_{\Gamma_D} = g_D \right\}$$

$$v \in H^1_0(\Omega) = \left\{ v \in H^1(\Omega) \;\middle|\; v|_{\Gamma_D} = 0 \right\}$$

(2) bilinear and linear forms

$$a(u,v) = \int_\Omega \left[A\nabla u \cdot \nabla v + a u v \right] dx$$

$$f(v) = \int_\Omega f v \, dx + \int_{\Gamma_N} g_N v \, ds$$

(VF) Find $u \in H^1_g(\Omega)$ s.t.

$$a(u,v) = f(v) \quad \forall v \in H^1_0(\Omega).$$

(3) boundary conditions

Dirichlet B.C. $\quad u|_{\Gamma_D} = g_D \quad$ essential B.C.

Neumann B.C. $\quad \vec{n} \cdot (A\nabla u)|_{\Gamma_N} = g_N \quad$ natural B.C.

- **Minimization Formulation**

Find $u \in H^1_g(\Omega)$ s.t.

$$J(u) = \min_{v \in H^1_g(\Omega)} J(v),$$

where $J(v) = \frac{1}{2} a(v,v) - f(v)$.

- **Least-squares Formulations**

(1) the primitive LS formulation

LS functional

$$\mathcal{L}_1(v;\vec{f}) = \left\| -\nabla\cdot(A\nabla u) + a u - f \right\|^2_{0,\Omega} + \left\| \vec{n}\cdot(A\nabla v) \right\|^2_{\frac{1}{2},\Gamma_N} + \left\| v \right\|^2_{\frac{3}{2},\Gamma_D}$$

solution space $u \in H^2(\Omega) = \left\{ v \in H^1(\Omega) \;\middle|\; \nabla v \in H^1(\Omega)^d \right\}$

Find $u \in H^2(\Omega)$ s.t.

$$\mathcal{L}_1(u;\vec{f}) = \min_{v \in H^2(\Omega)} \mathcal{L}_1(v;\vec{f})$$

(2) first-order system LS formulation

first-order system

$$\begin{cases} \vec{\sigma} + A\nabla u = \vec{0}, \\ -\nabla\cdot\vec{\sigma} + au = f, \end{cases} \quad \text{in } \Omega \qquad \begin{cases} u\big|_{\Gamma_D} = g_D \\ \vec{n}\cdot\vec{\sigma}\big|_{\Gamma_N} = g_N \end{cases}$$

LS functional

$$\mathcal{L}_2\left(\vec{\tau}, v; \vec{f}\right) = \left\| A^{-\frac{1}{2}}\left(\vec{\tau} + A\nabla v\right)\right\|_{0,\Omega}^2 + \left\| -\nabla\cdot\vec{\tau} + av - f\right\|_{0,\Omega}^2$$

$$+ \|v\|_{\frac{1}{2},\Gamma_D}^2 + \|\vec{n}\cdot\vec{\tau}\|_{-\frac{1}{2},\Gamma_N}^2$$

solution spaces

$$u \in H^1(\Omega) \quad \text{and} \quad \vec{\sigma} \in H(\mathrm{div};\Omega) = \left\{ \vec{\tau} \in L^2(\Omega)^d \,\middle|\, \nabla\cdot\vec{\tau} \in L^2(\Omega) \right\}$$

Find $(\vec{\sigma}, u) \in H(\mathrm{div};\Omega) \times H^1(\Omega)$ s.t.

$$\mathcal{L}_2\left(\vec{\sigma}, u; \vec{f}\right) = \min_{(\vec{\tau}, v)\in H(\mathrm{div})\times H^1} \mathcal{L}_2\left(\vec{\tau}, v; \vec{f}\right)$$

- Mathmatical Questions

(1) well-posedness of the problem

existence :

uniqueness :

stability :

(2) relations between formulations